

# AN ERDŐS-KO-RADO THEOREM FOR FINITE 2-TRANSITIVE GROUPS

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**ABSTRACT.** We prove an analogue of the classical Erdős-Ko-Rado theorem for intersecting sets of permutations in finite 2-transitive groups. Given a finite group  $G$  acting faithfully and 2-transitively on the set  $\Omega$ , we show that an intersecting set of maximal size in  $G$  has cardinality  $|G|/|\Omega|$ . This generalises and gives a unifying proof of some similar recent results in the literature.

## 1. GENERAL RESULTS

The Erdős-Ko-Rado theorem [14] determines the cardinality and also describes the structure of a set of maximal size of intersecting  $k$ -subsets from  $\{1, \dots, n\}$ . The theorem shows that provided that  $n > 2k$ , a set of maximal size of intersecting  $k$ -subsets from  $\{1, \dots, n\}$  has cardinality  $\binom{n-1}{k-1}$  and is the set of all  $k$ -subsets that contain a common fixed element. (For our work it is useful to emphasize that this theorem consists of two distinct parts: the first part determines the maximal size of intersecting  $k$ -subsets; the second part classifies the sets attaining this maximum.) Analogous results hold for many other combinatorial and algebraic objects other than sets, and in this paper we are concerned with an extension of the Erdős-Ko-Rado theorem to permutation groups.

Let  $G$  be a permutation group on  $\Omega$ . A subset  $S$  of  $G$  is said to be *intersecting* if, for every  $g, h \in S$ , the permutation  $gh^{-1}$  fixes some point of  $\Omega$  (note that this implies that  $\alpha^g = \alpha^h$ , for some  $\alpha \in \Omega$ ). As with the Erdős-Ko-Rado theorem, in this context we are interested in finding the cardinality of an intersecting set of maximal size in  $G$  and possibly classifying the sets that attain this bound.

The main theorem of this paper answers the first question for 2-transitive groups.

**Theorem 1.1.** *Let  $G$  be a finite 2-transitive group on the set  $\Omega$ . An intersecting set of maximal size in  $G$  has cardinality  $|G|/|\Omega|$ .*

Before giving some specific comments on Theorem 1.1 (which ultimately relies on the classification of finite simple groups and on some detailed analysis of the representation theory of some Lie type groups), we give some historical background on this area of research.

**1.1. Erdős-Ko-Rado-type theorems for permutation groups.** Possibly the most interesting permutation group and the most intriguing combinatorial object is the finite symmetric group  $\text{Sym}(n)$  of degree  $n$ . Here, the natural extension of the Erdős-Ko-Rado theorem for  $\text{Sym}(n)$  was independently proved in [8] and [27]. These papers, using different methods, showed that every intersecting set of  $\text{Sym}(n)$  has cardinality at most the ratio  $|\text{Sym}(n)|/|\{1, \dots, n\}| = (n-1)!$ . They both further showed that the only intersecting sets meeting this bound are the cosets of the

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stabiliser of a point. The same result was also proved in [21] using the character theory of  $\text{Sym}(n)$ .

Despite the exact characterization of the largest intersecting sets for  $\text{Sym}(n)$ , Theorem 1.1 cannot be strengthened to include such a characterization for all 2-transitive groups. There are various 2-transitive groups having intersecting sets of size  $|G|/|\Omega|$  which are not the cosets of the stabiliser of a point, see Conjecture 1.2 below or [28, 29] for some examples. Currently, it is not clear to what extent (that is, for which families of permutation groups) the complete analogue of the Erdős-Ko-Rado theorem holds.

Recently there have been many papers proving that the natural extension of the Erdős-Ko-Rado theorem holds for specific permutation groups  $G$  (see [1, 13, 26, 28, 29, 33]) and there are also two papers, [2] and [3], that consider when the natural extension of the Erdős-Ko-Rado theorem holds for transitive and 2-transitive groups. Again, this means asking if the largest intersecting sets in  $G$  are the cosets in  $G$  of the stabiliser of a point. Typically, a permutation group may have intersecting sets of size larger than the size of the stabiliser of a point, let alone hope that every such intersecting set is the coset of the stabiliser of a point. However, a behaviour very similar to  $\text{Sym}(n)$  is offered by  $\text{PGL}_2(q)$  in its natural action on the projective line [28, Theorem 1]; the intersecting sets of maximal size in  $\text{PGL}_2(q)$  are exactly the cosets of the stabiliser of a point. It is not hard to see that in projective general linear groups of dimension greater than 2 there are maximum intersecting sets that are not the cosets of the stabiliser of a point (for instance, the cosets of the stabiliser of a hyperplane). This lead the first and the second author to pose the following conjecture:

**Conjecture 1.2** ([28, Conjecture 2]). *The intersecting sets of maximal size in  $\text{PGL}_n(q)$  acting on the points of the projective space are exactly the cosets of the stabiliser of a point and the cosets of the stabiliser of a hyperplane.*

This conjecture has been settled only for  $n \in \{1, 2\}$  in [28, 29].

**1.2. A conjecture inspired by Theorem 1.1.** As far as we are aware, Theorem 1.1 is the first result that holds for a very rich family of permutation groups; thus far, most Erdős-Ko-Rado-type of theorems for permutation groups have been proved for rather specific families. In this paper we aim for a more general result.

For some 2-transitive groups we prove a stronger version of Theorem 1.1, which we now explain. Let  $V$  be the vector space over  $\mathbb{C}$  having a basis  $(g)_{g \in G}$  indexed by the elements of  $G$ , thus  $V$  is the underlying vector space of the group algebra  $\mathbb{C}G$ . Given a subset  $S$  of  $G$ , we write  $\chi_S = \sum_{s \in S} s$  for the characteristic vector of  $S$ . Set

$$W = \langle \chi_S \mid S \text{ coset of the stabiliser of a point in } G \rangle.$$

For some 2-transitive groups  $G$ , we show that if  $S$  is an intersecting set of maximal size, then  $\chi_S \in W$ . We actually dare to state the following conjecture (which we like to think as the algebraic analogue of the combinatorial Erdős-Ko-Rado theorem).

**Conjecture 1.3.** *Let  $G$  be a finite 2-transitive group and let  $V$  be the subspace of the group algebra  $\mathbb{C}G$  spanned by the characteristic vectors of the cosets of the point stabilisers. If  $S$  is an intersecting set of maximal cardinality in  $G$ , then the characteristic vector of  $S$  lies in  $V$ .*

**1.3. Comments on Theorem 1.1 and structure of the paper.** The proof of Theorem 1.1 depends upon the classification of the finite 2-transitive groups and hence on the classification of the finite simple groups.

In Section 2, we show that the problem of determining the intersecting sets of a transitive group  $G$  is equivalent to the problem of determining the independent sets in the Cayley graph over  $G$  with connection set the derangements of  $G$ . We then show some elementary results showing that the latter problem is related to the character theory of  $G$  and we prove some lemmas that will be useful when  $G$  is 2-transitive.

In Section 3, we reduce the proof of Theorem 1.1 to the case that  $G$  is one of the following non-abelian simple groups:  $\text{PSL}_n(q)$ ,  $\text{Sp}_{2n}(2)$ ,  $\text{PSU}_3(q)$ , Suzuki groups  $\text{Sz}(q)$ , Ree groups  $\text{Ree}(q)$  and the sporadic Higman-Sims group  $HS$ . The proof of Theorem 1.1 for  $\text{Sz}(q)$ ,  $\text{Ree}(q)$ ,  $HS$  and  $\text{PSU}_3(q)$  is in Section 4, 5, 6 and 7 respectively and depends heavily on the character table of these groups (unitary groups are by far the hardest case here). Finally the proof of Theorem 1.1 for  $\text{PSL}_n(q)$  and  $\text{Sp}_{2n}(2)$  is in Section 8 and 9 respectively and depends on the theory of Weil representations of these groups.

## 2. SOME ALGEBRAIC GRAPH THEORY

The problem of determining the intersecting sets of a permutation group  $G$  can be formulated in graph-theoretic terminology. We denote by  $\Gamma_G$  the *derangement graph* of  $G$ : the vertices of this graph are the elements of  $G$  and the edges are the unordered pairs  $\{g, h\}$  such that  $gh^{-1}$  is a *derangement*, that is,  $gh^{-1}$  fixes no point. Now, an intersecting set of  $G$  is simply an *independent set* or a *clique* of  $\Gamma_G$ , and (similarly) the classical Erdős-Ko-Rado theorem translates into a classification of the independent sets of maximal cardinality of the Kneser graphs.

Since the right regular representation of  $G$  is a subgroup of the automorphism group of  $\Gamma_G$ , we see that  $\Gamma_G$  is a *Cayley graph*. Namely, if  $\mathcal{D}$  is the set of derangements of  $G$ , then  $\Gamma_G$  is the Cayley graph on  $G$  with connection set  $\mathcal{D}$ . Clearly,  $\mathcal{D}$  is a union of  $G$ -conjugacy classes, so  $\Gamma_G$  is a *normal* Cayley graph.

As usual, we simply say that the complex number  $\xi$  is an *eigenvalue* of the graph  $\Gamma$  if  $\xi$  is an eigenvalue of the adjacency matrix of  $\Gamma$ . We use  $\text{Irr}(G)$  to denote the *irreducible complex characters* of the group  $G$  and given  $\chi \in \text{Irr}(G)$  and a subset  $S$  of  $G$  we write

$$\chi(S) = \sum_{s \in S} \chi(s).$$

In the following lemma we recall that the eigenvalues of a normal Cayley graph on  $G$  are determined by the irreducible complex characters of  $G$ . This a well-known result, we refer the reader to Babai's work [4] and a proof is also given in [20, Section 11.12].

**Lemma 2.1.** *Let  $G$  be a permutation group on  $\Omega$  and let  $\mathcal{D}$  be the set of derangements of  $G$ . The spectrum of the graph  $\Gamma_G$  is  $\{\chi(\mathcal{D})/\chi(1) \mid \chi \in \text{Irr}(G)\}$ . Also, if  $\tau$  is an eigenvalue of  $\Gamma_G$  and  $\chi_1, \dots, \chi_s$  are the irreducible characters of  $G$  such that  $\tau = \chi_i(\mathcal{D})/\chi_i(1)$ , then the dimension of the  $\tau$ -eigenspace of  $\Gamma_G$  is  $\sum_{i=1}^s \chi_i(1)^2$ .*

For simplicity, we will write  $\lambda(\chi)$  for the eigenvalue of  $\Gamma_G$  afforded by the irreducible character  $\chi$ , that is,

$$(2.1) \quad \lambda(\chi) = \frac{\chi(\mathcal{D})}{\chi(1)} = \frac{1}{\chi(1)} \sum_{g \in \mathcal{D}} \chi(g).$$

The next result is the well-known *ratio-bound* for independent sets in a graph; for a proof tailored to our needs see for example [28, Lemma 3].

**Lemma 2.2.** *Let  $G$  be a permutation group, let  $\tau$  be the minimum eigenvalue of  $\Gamma_G$ , let  $d$  be the valency of  $\Gamma_G$  and let  $S$  be an independent set of  $\Gamma_G$ . Then*

$$\frac{|S|}{|G|} \leq \left(1 - \frac{d}{\tau}\right)^{-1}.$$

*If the equality is met, then  $\chi_S - \frac{|S|}{|G|}\chi_G$  is an eigenvector of  $\Gamma_G$  with eigenvalue  $\tau$ .*

At this point it is worthwhile to give a hint on (a simplified version of) one of our main strategies in proving Theorem 1.1. Let  $G$  be a 2-transitive group on  $\Omega$  and let  $\pi$  be its permutation character. Write  $\pi = \chi_0 + \psi$ , where  $\chi_0$  is the principal character of  $G$ . Observe that  $\psi$  is an irreducible character of  $G$  because  $G$  is 2-transitive.

We will show that in many cases the minimal eigenvalue of  $\Gamma_G$  is afforded by the irreducible complex character  $\psi$ , that is, the minimal eigenvalue of  $\Gamma_G$  is  $\lambda(\psi)$ , and hence this minimum is

$$\lambda(\psi) = \frac{1}{\psi(1)} \sum_{g \in \mathcal{D}} \psi(g) = \frac{1}{|\Omega| - 1} \sum_{g \in \mathcal{D}} -1 = -\frac{|\mathcal{D}|}{|\Omega| - 1}.$$

In this case, using Lemma 2.2, we get that an independent set of  $\Gamma_G$  has cardinality at most

$$|G| \left(1 - \frac{|\mathcal{D}|}{\lambda(\psi)}\right)^{-1} = |G| \left(1 - \frac{|\mathcal{D}|}{-|\mathcal{D}|/(|\Omega| - 1)}\right)^{-1} = \frac{|G|}{|\Omega|}$$

and hence Theorem 1.1 holds for  $G$ .

Actually, sometimes a slightly stronger version of this strategy works, which allows us to prove Conjecture 1.3 for these cases. In fact, assume that the minimal eigenvalue of  $\Gamma_G$  is  $-|\mathcal{D}|/(|\Omega| - 1)$  and that  $\psi$  is the unique irreducible character of  $G$  affording this eigenvalue. Then, combining Lemmas 2.1 and 2.2, we see that

$$W = \langle \chi_S \mid S \text{ an independent set of } \Gamma_G \text{ of cardinality } |G|/|\Omega| \rangle$$

is a subspace of  $\mathbb{C}G$  with dimension  $\chi_0(1)^2 + \psi(1)^2 = 1 + (|\Omega| - 1)^2$ . Now, [28, Proposition 3.2] and also [2] show that

$$V = \langle \chi_S \mid S \text{ coset of the stabiliser of a point} \rangle$$

has also dimension  $1 + (|\Omega| - 1)^2$ . Since  $W \leq V$ , we get  $W = V$  and hence Conjecture 1.3 holds for  $G$ .

For future reference we highlight the argument in the previous paragraph in the following:

**Lemma 2.3.** *Let  $G$  be a 2-transitive group on  $\Omega$  and let  $\mathcal{D}$  be the set of derangements of  $G$ . If the minimum eigenvalue of  $\Gamma_G$  is  $-|\mathcal{D}|/(|\Omega| - 1)$  and if  $G$  has a unique irreducible character realising this minimum, then both Theorem 1.1 and Conjecture 1.3 hold for  $G$ .*

The following will prove useful in various occasions.

**Lemma 2.4.** *Let  $G$  be a 2-transitive group on  $\Omega$ , let  $\mathcal{D}$  be the set of derangements of  $G$  and let  $\pi = \chi_0 + \psi$  be the permutation character of  $G$  with  $\chi_0$  the principal character of  $G$ . Let  $\chi^*$  be an irreducible complex character of  $G$  with  $\chi^* \neq \psi$ . If  $\lambda(\chi^*)$  is the minimum eigenvalue of  $\Gamma_G$ , then*

$$\chi^*(1) \leq (|\Omega| - 1) \sqrt{\frac{|G|}{|\mathcal{D}|}} - 2.$$

*Proof.* Let  $A$  be the adjacency matrix of  $\Gamma_G$ , so  $A$  is the  $|G| \times |G|$ -matrix with rows and columns labelled by the elements of  $G$  defined by

$$A_{g,h} = \begin{cases} 1 & \text{if } gh^{-1} \in \mathcal{D}, \\ 0 & \text{if } gh^{-1} \notin \mathcal{D}. \end{cases}$$

Given a square matrix  $X = (x_{i,j})_{i,j}$ , we denote by  $\text{Tr}(X) = \sum_i x_{i,i}$  the trace of  $X$ . A standard counting argument gives

$$\begin{aligned} \text{Tr}(A^2) &= \sum_{g \in G} (A^2)_{g,g} = \sum_{g \in G} \sum_{h \in G} A_{g,h} A_{h,g} \\ &= \sum_{g \in G} |\{h \in G \mid gh^{-1} \in \mathcal{D}\}| = \sum_{g \in G} |\mathcal{D}| = |G||\mathcal{D}|. \end{aligned}$$

Now, the matrix  $A$  (and hence  $A^2$ ) can be diagonalised over  $\mathbb{C}$  and it follows immediately from Lemma 2.1 that

$$(2.2) \quad |G||\mathcal{D}| = \text{Tr}(A^2) = \sum_{\chi \in \text{Irr}(G)} \chi(1)^2 \left( \frac{\chi(\mathcal{D})}{\chi(1)} \right)^2 = \sum_{\chi \in \text{Irr}(G)} \chi(\mathcal{D})^2.$$

Since  $\lambda(\chi^*)$  is the minimum eigenvalue of  $\Gamma_G$ , we have

$$\lambda(\chi^*) \leq \lambda(\psi) = -|\mathcal{D}|/(|\Omega| - 1) < 0$$

and hence  $\chi^* \neq \chi_0$ .

All the terms in the summation in the right-hand side of Eq. (2.2) are positive, so we get that  $|G||\mathcal{D}| \geq \chi_0(\mathcal{D})^2 + \psi(\mathcal{D})^2 + \chi^*(\mathcal{D})^2$ . Since  $\chi_0(\mathcal{D}) = -\psi(\mathcal{D}) = |\mathcal{D}|$ , this becomes  $\chi^*(\mathcal{D})^2 \leq |G||\mathcal{D}| - 2|\mathcal{D}|^2$ . Thus

$$\begin{aligned} (2.3) \quad |\lambda(\chi^*)| &= \frac{|\chi^*(\mathcal{D})|}{\chi^*(1)} \leq \frac{|\mathcal{D}|}{\chi^*(1)} \sqrt{\frac{|G|}{|\mathcal{D}|} - 2} = \frac{|\mathcal{D}|}{\psi(1)} \frac{\psi(1)}{\chi^*(1)} \sqrt{\frac{|G|}{|\mathcal{D}|} - 2} \\ &= |\lambda(\psi)| \frac{\psi(1)}{\chi^*(1)} \sqrt{\frac{|G|}{|\mathcal{D}|} - 2} = |\lambda(\psi)| \frac{|\Omega| - 1}{\chi^*(1)} \sqrt{\frac{|G|}{|\mathcal{D}|} - 2}. \end{aligned}$$

Now,  $\lambda(\chi^*) \leq \lambda(\psi)$  and hence  $|\lambda(\chi^*)| \geq |\lambda(\psi)|$ . Thus the proof follows from Eq. (2.3).  $\square$

Incidentally, the proof of Lemma 2.4 shows that  $|G|/|\mathcal{D}| \geq 2$  and hence at most half of the elements of a 2-transitive group are derangements, this observation is a special case of the main result in [25].

Some suspicious reader, misguided by the comments preceding Lemma 2.3, might think that the hypothesis of Lemma 2.4 are never satisfied. In fact, by wishful thinking, one tends to hope that  $\lambda(\psi) = -|\mathcal{D}|/(|\Omega| - 1)$  is always the minimum eigenvalue of  $\Gamma_G$ . However this is not the case. The simplest example is given by the 2-transitive action of the Higman-Sims group  $HS$  of degree 176: here  $\psi(1) = 176 - 1 = 175$  and  $\lambda(\psi) = -79806$ , but there exists an irreducible character  $\chi$  of degree 22 with  $\lambda(\chi) = -118650$ .

In Sections 6, 7 and 9, the Higman-Sims group,  $\text{PSU}_3(q)$  and  $\text{Sp}_{2n}(2)$  are dealt with using a generalised version of Lemma 2.2, which we now discuss. Given a graph  $\Gamma$ , a square matrix  $A$  with rows and columns indexed by the vertices of  $\Gamma$  is said to be a *weighted adjacency matrix* of  $\Gamma$  if  $A_{u,v} = 0$  whenever  $u$  and  $v$  are not adjacent vertices. The ratio-bound (a.k.a. Lemma 2.2) holds not just for the adjacency matrix, but also for a weighted adjacency matrix (see [20, Section 2.4] for a proof).

**Lemma 2.5.** *Let  $G$  be a permutation group and let  $A$  be a weighted adjacency matrix of  $\Gamma_G$ . Let  $d$  be the largest eigenvalue and let  $\tau$  be the least eigenvalue of  $A$ . If  $S$  is an independent set in  $\Gamma_G$ , then*

$$\frac{|S|}{|G|} \leq \left(1 - \frac{d}{\tau}\right)^{-1}.$$

For the scope of this paper, we only consider very special types of weighted adjacency matrices. Let  $G$  be a permutation group, let  $\mathcal{D}$  be the set of derangements of  $G$  and let  $\mathcal{C}_1, \dots, \mathcal{C}_\ell$  be the  $G$ -conjugacy classes contained in  $\mathcal{D}$ . Given  $\mathbf{a} = (a_1, \dots, a_\ell) \in \mathbb{C}^\ell$ , define the  $\mathbf{a}$ -weighted adjacency matrix of  $\Gamma_G$  to be the matrix  $A$  with rows and columns indexed by elements of  $G$  and

$$A_{\sigma, \pi} = \begin{cases} a_i & \text{if } \sigma^{-1}\pi \in \mathcal{C}_i, \\ 0 & \text{otherwise.} \end{cases}$$

It is implicit in the work of Babai in [4] (see also [20, Section 11.12]) that the eigenvalues of  $A$  are determined by the irreducible complex characters of  $G$ . In fact, as in Eq. (2.1), given an irreducible character  $\chi$  of  $G$ , the eigenvalue of the  $\mathbf{a}$ -weighted adjacency matrix determined by  $\chi$  is

$$(2.4) \quad \lambda(\chi, \mathbf{a}) = \frac{1}{\chi(1)} \sum_{i=1}^{\ell} \left( a_i \sum_{x \in \mathcal{C}_i} \chi(x) \right) = \frac{1}{\chi(1)} \sum_{i=1}^{\ell} a_i |\mathcal{C}_i| \chi(x_i),$$

where  $(x_i)_{i \in \{1, \dots, \ell\}}$  is a set of representatives of the  $G$ -conjugacy classes  $(\mathcal{C}_i)_{i \in \{1, \dots, \ell\}}$ . Clearly, when  $\mathbf{a} = (1, \dots, 1)$ , we recover the eigenvalues of the adjacency matrix of  $\Gamma_G$ .

For later reference it is worth to point out that, if the weights  $a_i$  are non-negative real numbers, then the principal character will give the largest eigenvalue of the  $\mathbf{a}$ -weighted adjacency matrix. (This fact follows easily from Eq. (2.4) and from the inequality  $|\chi(g)| \leq \chi(1)$ , which is valid for every  $g \in G$  and for every irreducible character  $\chi$ .)

We conclude this section proving an analogue of Lemma 2.4.

**Lemma 2.6.** *Let  $G$  be a 2-transitive group on  $\Omega$ , let  $\mathcal{D}$  be the set of derangements of  $G$ , let  $\mathcal{C}_1, \dots, \mathcal{C}_\ell$  be the  $G$ -conjugacy classes contained in  $\mathcal{D}$  and let  $\pi = \chi_0 + \psi$  be the permutation character of  $G$  with  $\chi_0$  the principal character of  $G$ .*

*Let  $a_1, \dots, a_\ell$  be non-negative real numbers not all equal to zero and set  $\mathbf{a} = (a_1, \dots, a_\ell)$ , and let  $A$  be the  $\mathbf{a}$ -adjacency matrix of  $\Gamma_G$ .*

*If  $\lambda(\chi^*, \mathbf{a})$  is the minimum eigenvalue of  $A$  for some irreducible character  $\chi^*$  with  $\chi^* \neq \psi$ , then*

$$\chi^*(1) \leq (|\Omega| - 1) \sqrt{|G| \frac{\sum_{i=0}^{\ell} a_i^2 |\mathcal{C}_i|}{(\sum_{i=1}^{\ell} a_i |\mathcal{C}_i|)^2}} - 2.$$

*Proof.* The proof is similar to the proof of Lemma 2.4 and hence we omit most of the details. Eq. (2.4) yields

$$\lambda(\chi_0, \mathbf{a}) = \sum_{i=1}^{\ell} a_i |\mathcal{C}_i|, \quad \lambda(\psi, \mathbf{a}) = \frac{-1}{|\Omega| - 1} \sum_{i=1}^{\ell} a_i |\mathcal{C}_i|.$$

Similar to the proof of Lemma 2.4,

$$\text{Tr}(A^2) = |G| \sum_{i=0}^{\ell} a_i^2 |\mathcal{C}_i|.$$

The trace of  $A^2$  is also equal to the sum of the squares of the eigenvalues of  $A$ . Thus, just as in the proof of Lemma 2.4, this implies that

$$\chi^*(1)^2 \lambda(\chi^*, \mathbf{a})^2 \leq |G| \sum_{i=0}^{\ell} a_i^2 |\mathcal{C}_i| - 2 \left( \sum_{i=1}^{\ell} a_i |\mathcal{C}_i| \right)^2.$$

Since  $\lambda(\chi^*, \mathbf{a})$  is the minimum eigenvalue of  $A$ , we have  $\lambda(\chi^*, \mathbf{a}) \leq \lambda(\psi, \mathbf{a}) < 0$  and hence  $|\lambda(\psi, \bar{a})| \leq |\lambda(\chi^*, \mathbf{a})|$ . Now, following the last paragraph of the proof of Lemma 2.4, we obtain

$$\chi^*(1) \leq (|\Omega| - 1) \sqrt{|G| \frac{\sum_{i=0}^{\ell} a_i^2 |\mathcal{C}_i|}{(\sum_{i=1}^{\ell} a_i |\mathcal{C}_i|)^2} - 2}.$$

□

### 3. THREE REDUCTIONS

Let  $G$  be a finite 2-transitive group on  $\Omega$ . A 1911 celebrated theorem of Burnside shows that either  $G$  contains a regular subgroup or  $G$  is almost simple, see [12, Theorem 4.1B]. In the first case,  $G$  contains a subgroup  $H$  with  $H$  transitive on  $\Omega$  and with every non-identity element of  $H$  being a derangement on  $\Omega$ . In particular,  $H$  is a clique of cardinality  $|\Omega|$  in the derangement graph  $\Gamma_G$ . It is then relevant for Theorem 1.1 the following well-known lemma (usually referred to as the clique-coclique bound, see for example [20, Section 2.1]).

**Lemma 3.1.** *Let  $\Gamma$  be a finite graph of order  $n$  having a group of automorphisms acting transitively on its vertices. Let  $C$  be a clique of  $\Gamma$  and let  $S$  be a coclique of  $\Gamma$ . Then  $|C||S| \leq n$  with equality if and only if  $|C \cap S^g| = 1$  for every  $g \in G$ .*

In particular, Theorem 1.1 follows immediately from Lemma 3.1 for 2-transitive groups having a regular subgroup. Therefore, for the rest of this paper, we assume that  $G$  is almost simple, that is,  $G$  contains a non-abelian simple normal subgroup  $S$  and  $G$  acts faithfully by conjugation on  $S$ . Thus, identifying  $S$  as a subgroup of  $\text{Aut}(S)$ , we have

$$S \leq G \leq \text{Aut}(S),$$

for some non-abelian simple group  $S$ . This is our first reduction.

The classification of finite simple groups has allowed for the complete classification of the finite 2-transitive groups. For the reader's convenience we report in Table 1 such classification extracted from [7, page 197].

An easy computation with a computer (using the computer algebra system **Magma** [5]) allows us to show that Theorem 1.1 holds true for many of the lines in Table 1.

**Proposition 3.2.** *Let  $G$  be a 2-transitive group as in lines 1, 8–12, or 14 of Table 1. Then Theorem 1.1 holds for  $G$ .*

*Proof.* If  $G$  is as in line 1, then  $G = \text{Alt}(n)$  or  $G = \text{Sym}(n)$ . In both cases from [8, 21, 26, 27], every intersecting set of maximal size of  $G$  is the coset of the stabiliser of a point and hence the lemma follows immediately.

Now for every group  $G$  as in lines 8–12 or 14 of Table 1 we may compute the minimum eigenvalue of  $\Gamma_G$  using Lemma 2.1 and the computer algebra system **Magma** (or the character tables in [11]). In all cases, we see that the minimum eigenvalue is  $-|\mathcal{D}|/(|\Omega| - 1)$ , where  $\mathcal{D}$  is the set of derangements of  $G$  and  $\Omega$  is the set acted upon by  $G$ . Thus the proposition follows from Lemma 2.2.

□

Line	Group $S$	Degree	Condition on $G$	Remarks
1	$\text{Alt}(n)$	$n$	$\text{Alt}(n) \leq G \leq \text{Sym}(n)$	$n \geq 5$
2	$\text{PSL}_n(q)$	$\frac{q^n-1}{q-1}$	$\text{PSL}_n(q) \leq G \leq \text{P}\Gamma\text{L}_n(q)$	$n \geq 2, (n, q) \neq (2, 2), (2, 3)$
3	$\text{Sp}_{2n}(2)$	$2^{n-1}(2^n-1)$	$G = S$	$n \geq 3$
4	$\text{Sp}_{2n}(2)$	$2^{n-1}(2^n+1)$	$G = S$	$n \geq 3$
5	$\text{PSU}_3(q)$	$q^3+1$	$\text{PSU}_3(q) \leq G \leq \text{P}\Gamma\text{U}_3(q)$	$q \neq 2$
6	$\text{Sz}(q)$	$q^2+1$	$\text{Sz}(q) \leq G \leq \text{Aut}(\text{Sz}(q))$	$q = 2^{2m+1}, m > 0$
7	$\text{Ree}(q)$	$q^3+1$	$\text{Ree}(q) \leq G \leq \text{Aut}(\text{Ree}(q))$	$q = 3^{2m+1}, m > 0$
8	$M_n$	$n$	$M_n \leq G \leq \text{Aut}(M_n)$	$n \in \{11, 12, 22, 23, 24\}$ , $M_n$ Mathieu group, $G = S$ or $n = 22$
9	$M_{11}$	12	$G = S$	
10	$\text{PSL}_2(11)$	11	$G = S$	
11	$\text{Alt}(7)$	15	$G = S$	
12	$\text{PSL}_2(8)$	28	$G = \text{P}\Sigma\text{L}_2(8)$	
13	$HS$	176	$G = S$	$HS$ Higman-Sims group
14	$Co_3$	276	$G = S$	$Co_3$ third Conway group

TABLE 1. Finite 2-transitive groups of almost simple type

In particular, Proposition 3.2 is our second reduction and allows us to consider only lines 2, 3, 4, 5, 6, 7 and 13 of Table 1.

**Proposition 3.3.** *Let  $G$  be a transitive group on  $\Omega$  and let  $H$  be a transitive subgroup of  $G$ . Suppose that every intersecting set of  $H$  has cardinality at most  $|H|/|\Omega|$ , then every intersecting set of  $G$  has cardinality at most  $|G|/|\Omega|$ .*

*Proof.* Let  $X$  be an intersecting set of maximal cardinality of  $G$ . Let  $R$  be a set of representatives for the right cosets of  $H$  in  $G$ . Thus  $G = \cup_{r \in R} Hr$  and  $X = \cup_{r \in R} (Hr \cap X)$ . Choose  $\bar{r} \in R$  with  $|H\bar{r} \cap X|$  as large as possible. Observe that

$$(H\bar{r} \cap X)\bar{r}^{-1} = H \cap X\bar{r}^{-1}$$

is an intersecting set of  $H$ , and hence

$$|H\bar{r} \cap X| = |H \cap X\bar{r}^{-1}| \leq |H|/|\Omega|.$$

In particular,

$$|X| = \sum_{r \in R} |Hr \cap X| \leq |R||H\bar{r} \cap X| \leq |R||H|/|\Omega| = |G|/|\Omega|.$$

□

Proposition 3.3 offers our third reduction: in proving Theorem 1.1 we may assume that  $G = S$  is a non-abelian simple group. (In fact, in view of our first reduction  $G$  is almost simple with socle the non-abelian simple group  $S$ . Then, by Table 1, we see that either  $S$  itself is 2-transitive, or  $S = \text{PSL}_2(8)$  and  $G = \text{P}\Sigma\text{L}_2(8)$ . However we have considered the latter possibility in Proposition 3.2.)

#### 4. SUZUKI GROUPS: LINE 6 OF TABLE 1

In this section we follow the notation and the information on the Suzuki groups in [31]. Let  $\ell$  be a positive integer,  $q = 2^{2\ell+1}$  and  $r = 2^{\ell+1}$ . Observe that  $2q = r^2$ . We denote by  $\text{Sz}(q)$  the Suzuki group defined over the finite field with  $q$  elements. The group  $\text{Sz}(q)$  has order  $(q^2+1)q^2(q-1)$  and has a 2-transitive action on the points of an inversive plane of cardinality  $q^2+1$ . Let  $\pi$  be the permutation character of  $\text{Sz}(q)$  and write  $\pi = \chi_0 + X$  where  $\chi_0$  is the principal character of  $\text{Sz}(q)$  and  $X$



is an irreducible character of degree  $q^2$ . The conjugacy classes and the character table of  $\text{Sz}(q)$  are described in [31, Section 17].

Suzuki in [31] subdivides the irreducible complex characters into six families:

- (1): the principal character  $\chi_0$ ,
- (2): the character  $X$ ,
- (3): the family  $(X_i)_{i=1}^{q/2-1}$  of  $q/2 - 1$  characters of degree  $q^2 + 1$ ,
- (4): the family  $(Y_j)_{j=1}^{(q+4)/4}$  of  $(q+r)/4$  characters of degree  $(q-r+1)(q-1)$ ,
- (5): the family  $(Z_k)_{k=1}^{(q-r)/4}$  of  $(q-r)/4$  characters of degree  $(q+r+1)(q-1)$ ,  
and
- (6): the family  $(W_1, W_2)$  of two characters of degree  $r(q-1)/2$ .

The only conjugacy classes of  $\text{Sz}(q)$  relevant for our work are the conjugacy classes consisting of derangements. In the work of Suzuki these classes are partitioned into two families:

- (1): the family  $\pi_1$  consisting of  $(q+r)/4$  conjugacy classes each of cardinality  $(q-r+1)q^2(q-1)$ , and
- (2): the family  $\pi_2$  consisting of  $(q-r)/4$  conjugacy classes each of cardinality  $(q+r+1)q^2(q-1)$ .

The character values are then described in [31, Section 17 and Theorem 13]. Using Lemma 2.1, this information and straightforward computations we obtain Table 2, which describes the eigenvalues (together with their multiplicities) of the derangement graph of  $\text{Sz}(q)$ .

Eigenvalue	Multiplicity	Comments
$\frac{q^3(q-1)^2}{2}$	1	Afforded by $\chi_0$
$-\frac{q(q-1)^2}{2}$	$q^4$	Afforded by $X$
0	$(q^2+1)^2 \frac{q-2}{2}$	Afforded by $(X_i)_{i=1}^{q/2-1}$
$q^2$	$\frac{q(q-1)^3(q+1)}{2}$	Afforded by $(Y_i)_{i=1}^{(q+r)/4}$ , $(Z_k)_{k=1}^{(q-r)/4}$ and $(W_l)_{l=1}^2$

TABLE 2. Eigenvalues and multiplicities for the derangement graph of  $\text{Sz}(q)$

We are now ready to prove Theorem 1.1 for the 2-transitive groups with a Suzuki group as the socle.

**Proposition 4.1.** *Let  $G$  be a 2-transitive group as in line 6 of Table 1. Then Theorem 1.1 holds for  $G$ .*

*Proof.* We use the notation that we established above and the reductions in Section 3. Let  $S$  be the socle of  $G$ . Thus  $G = S = \text{Sz}(q)$ . Then, from Table 2, the graph  $\Gamma_G$  has a unique negative eigenvalue and hence the proof follows immediately from Lemma 2.3. □

## 5. REE GROUPS: LINE 7 OF TABLE 1

The analysis in this section is similar to the analysis in Section 4. (All the information needed in this section can be found in [34].)

Let  $\ell$  be a positive integer,  $q = 3^{2\ell+1}$  and  $m = 3^\ell$ . We denote by  $\text{Ree}(q)$  the Ree group defined over the finite field with  $q$  elements. The group  $\text{Ree}(q)$  has order  $(q^3+1)q^3(q-1)$  and has a 2-transitive action of degree  $q^3+1$  on the points of a Steiner system. Let  $\pi$  be the permutation character of  $\text{Ree}(q)$  and write  $\pi = \xi_1 + \xi_3$  where  $\xi_1$  is the principal character of  $\text{Ree}(q)$  and  $\xi_3$  is an irreducible character of

degree  $q^3$ . The conjugacy classes and the character table of  $\text{Ree}(q)$  are described in [34].

Since we are only interested in derangements, we only say a few words on the conjugacy classes of  $\text{Ree}(q)$  consisting of derangements. Ward [34] subdivides the conjugacy classes of derangements into four families:

- (1): a family consisting of  $(q-3)/24$  conjugacy classes each having size  $(q^2 - q + 1)q^3(q-1)$  (denoted by  $(S^a)_a$ );
- (2): a family consisting of  $(q-3)/8$  conjugacy classes each having size  $(q^2 - q + 1)q^3(q-1)$  (denoted by  $(JS^a)_a$ );
- (3): a family consisting of  $(q-3m)/6$  conjugacy classes each having size  $(q+1+3m)q^3(q^2-1)$  (denoted by  $V$ );
- (4): a family consisting of  $(q+3m)/6$  conjugacy classes each having size  $(q+1-3m)q^3(q^2-1)$  (denoted by  $W$ ).

Ward [34] subdivides the irreducible complex characters into several families (namely,  $\{\xi_i\}_i$ ,  $\{\eta_r\}_r$ ,  $\{\eta'_r\}_r$ ,  $\{\eta_t\}_t$ ,  $\{\eta'_t\}_t$ ,  $\{\eta_i^-\}_i$  and  $\{\eta_i^+\}_i$ ). The value of some of these characters on derangements is quite tricky to extract from [34]: especially the irreducible characters defined “exceptional” by Ward (like  $\{\eta_t\}_t$  and  $\{\eta'_t\}_t$ ). Because of this technical difficulties, we do not compute all the eigenvalues of the derangement graph of  $\text{Ree}(q)$ , but we content ourselves to the eigenvalues  $\lambda(\xi_i)$ , where  $i \in \{1, \dots, 10\}$ . In fact, for these characters the value  $\lambda(\xi_i)$  can be easily computed by simply reading the character table on page 87 and 88 in [34] and by using the information on the conjugacy classes of derangements of  $\text{Ree}(q)$  that we outlined above (and by invoking Lemma 2.1). We sum up this information in Table 3.

Eigenvalue	Comment
$\frac{q^3(q-1)(q^3-2q^2-1)}{2}$	Afforded by $\xi_1$ , valency of $\Gamma_{\text{Ree}(q)}$
$-\frac{(q-1)(q^3-2q^2-1)}{2}$	Afforded by $\xi_3$
0	Afforded by $\xi_2$ and $\xi_4$
$3mq^2(-q+4m-1)$	Afforded by $\xi_5$ and $\xi_7$
$3mq^2(q+4m+1)$	Afforded by $\xi_6$ and $\xi_8$
$q^3$	Afforded by $\xi_9$ and $\xi_{10}$

TABLE 3. Some eigenvalues for the derangement graph of  $\text{Ree}(q)$

**Proposition 5.1.** *Let  $G$  be a 2-transitive group as in line 7 of Table 1. Then Theorem 1.1 holds for  $G$ .*

*Proof.* We use the notation and the reductions that we established above. Let  $S$  be the socle of  $G$ . Thus  $G = S = \text{Ree}(q)$ . We show that  $\lambda(\xi_3)$  is the minimal eigenvalue of  $\Gamma_G$  and that  $\xi_3$  is the unique irreducible character affording this eigenvalue, the proof will then follow immediately from Lemma 2.3.

We argue by contradiction and we assume that there exists  $\chi^* \in \text{Irr}(G)$  with  $\lambda(\chi^*) \leq \lambda(\xi_3)$  and  $\chi^* \neq \xi_3$ . From Lemma 2.4 and from Table 3, we have

$$(5.1) \quad \chi^*(1) \leq (|\Omega| - 1) \sqrt{\frac{|G|}{|\mathcal{D}|}} - 2 = q^3 \sqrt{\frac{4q^2 + 4}{q^3 - 2q^2 - 1}}.$$

By comparing the character degrees of  $\text{Ree}(q)$  (given in [34, page 87]) with Eq. (5.1), we get  $\chi^* \in \{\xi_2, \xi_5, \xi_6, \xi_7, \xi_8, \xi_9, \xi_{10}\}$ . Now, Table 3 yields  $\lambda(\chi^*) > \lambda(\xi_3)$ , a contradiction.  $\square$

## 6. HIGMAN-SIMS GROUP: LINE 13 OF TABLE 1

Let  $G$  be the Higman-Sims group: this is a sporadic group with 44352000 elements and a 2-transitive action on a geometry, known as “Higman’s Geometry”, that has 176 points. Let  $\pi = \chi_0 + \psi$  denote the permutation character for this action and let  $\chi_0$  be the principal character of  $G$ . As stated in Section 2, there is an irreducible representation of  $G$  for which the corresponding eigenvalue of the derangement graph is strictly less than  $\lambda(\psi)$ . In this case we prove that Theorem 1.1 still holds for  $G$  using a weighted adjacency matrix.

The Higman-Sims group has 5 conjugacy classes of derangements (the class 4B, 8A, 11A, 11B and 15A using the notation in [11]). Weight the two classes 11A and 11B with a 1 and all other classes with a 0. Now, the eigenvalues of the weighted adjacency matrix can be directly calculated. The largest eigenvalue is 8064000, and the least is  $-46080$ . By Lemma 2.5, if  $S$  is a coclique in  $\Gamma_G$ , then

$$|S| \leq \frac{|G|}{1 - \frac{-46080}{8064000}} = \frac{|G|}{176}.$$

## 7. PROJECTIVE SPECIAL UNITARY GROUPS: LINE 5 OF TABLE 1

Let  $G$  be the group  $\text{PSU}_3(q)$ . We follow the notation and use the character table for  $G$  that is given in [30]. This group has order  $q^3(q^3 + 1)(q^2 - 1)/d$  where  $d = \gcd(3, q + 1)$  and a 2-transitive action with order  $q^3 + 1$ . Let  $\pi = \chi_0 + \psi$  denote the permutation character of this action and  $\chi_0$  the principal character. The eigenvalue  $\lambda(\psi)$  is not the least eigenvalue for the adjacency matrix of the derangement graph, so we will consider a weighted adjacency matrix.

For  $\gcd(3, q + 1) = 1$ , there are two families of conjugacy classes in  $\text{PSU}_3(q)$  that are derangements,  $C_1$  and  $C_2$ :

- (1):  $C_1$  has  $(q^2 - q)/3$  classes each of size  $|G|/(q^2 - q + 1)$ , and
- (2):  $C_2$  has  $(q^2 - q)/6$  classes each of size  $|G|/(q + 1)^2$ .

For the values of  $q$  with  $\gcd(q + 1, 3) = 3$ , there are three families of conjugacy classes of derangements in  $\text{PSU}_3(q)$  which we denote by  $C_1$ ,  $C'_2$  and  $C''_2$ :

- (1):  $C_1$  contains  $(q^2 - q - 2)/9$  classes each of size  $3|G|/(q^2 - q + 1)$ ,
- (2):  $C'_2$  contains  $(q^2 - q - 2)/18$  classes of size  $3|G|/(q + 1)^2$  and
- (3):  $C''_2$  contains a single class of size  $|G|/(q + 1)^2$ .

Set  $C_2 = C'_2 \cup C''_2$ .

Define the  $(a, b)$ -weighted adjacency matrix  $A$  as follows. The  $(\sigma, \pi)$ -entry of  $A$  is equal to  $a$  if  $\sigma^{-1}\pi$  is in one of the conjugacy classes in the family  $C_1$ , and the entry is equal to  $b$  if  $\sigma^{-1}\pi$  is in one of the conjugacy classes in the family  $C_2$ , and any other entry is 0. Set

$$a = \frac{q(2q^2 + q - 1)}{3|C_1|}, \quad b = \frac{q(q^2 - q + 1)}{3|C_2|},$$

where (with an abuse of notation)  $|C_i|$  denotes the total number of elements in the conjugacy classes in the family  $C_i$  rather than the size of the family. Thus

$$\begin{aligned} |C_1| &= \begin{cases} \frac{q^4(q^2-1)^2}{3} & \text{if } \gcd(q+1, 3) = 1, \\ \frac{q^3(q-1)(q+1)^3(q-2)}{9} & \text{if } \gcd(q+1, 3) = 3, \end{cases} \\ |C_2| &= \begin{cases} \frac{q^4(q-1)^2(q^2-q+1)}{6} & \text{if } \gcd(q+1, 3) = 1, \\ \frac{q^3(q-1)(q^2-q+1)(q^2-q+4)}{18} & \text{if } \gcd(q+1, 3) = 3. \end{cases} \end{aligned}$$

Straight-forward calculations (using the table in [30]) produces the eigenvalues of  $A$  for three of the irreducible characters of  $\text{PSU}_3(q)$  (for all admissible values of  $q$ ).

**Lemma 7.1.** *The eigenvalue of  $A$  afforded by the principal character is  $q^3$ . The eigenvalue afforded by  $\psi$  is  $-1$ . The eigenvalue afforded by the irreducible representation with degree  $q(q-1)$  is  $-1$ .*

We will calculate the exact value of the eigenvalues of  $A$  corresponding the representations of degree  $q^2 - q + 1$ . We will consider the cases where  $\gcd(3, q+1) = 1$  and  $\gcd(3, q+1) = 3$  separately.

**Lemma 7.2.** *Assume that  $\gcd(3, q+1) = 1$ . If  $q$  is odd, then the eigenvalue for exactly one of the irreducible representations with degree  $q^2 - q + 1$  is equal to  $-1$ , for any other irreducible representations with degree  $q^2 - q + 1$ , the eigenvalue is equal to  $2/(q-1)$ . If  $q$  is even, then the eigenvalue for every irreducible representation of degree  $q^2 - q + 1$  is equal to  $2/(q-1)$ .*

*Proof.* Define

$$T = \{(k, l, m) : k + l + m \equiv 0 \pmod{q+1}, 1 \leq k < l < m \leq q+1\}.$$

For  $\gcd(3, q+1) = 1$ , the conjugacy classes in  $C_2$  are parametrised by triples from the set  $T$ . The irreducible representations with degree  $q^2 - q + 1$  are parametrised by  $u \in \{1, \dots, q\}$  and we will denote them by  $\chi_u$ . The value of  $\chi_u$  on the conjugacy classes in  $C_1$  is 0. The sum of the character  $\chi_u$  over all the conjugacy classes in the family  $C_2$  is

$$(7.1) \quad \sum_{(k,l,m) \in T} e^{3uk} + e^{3ul} + e^{3um},$$

where  $e$  is a complex primitive  $(q+1)$ th root of unity.

Simple counting arguments (that we omit) will show the following two results.

**Claim 7.3.** *If  $q$  is odd, then*

- (1): *the element  $q+1$  occurs in exactly  $(q-1)/2$  triples in  $T$ ,*
- (2): *any odd element from  $\{1, \dots, q\}$  occurs in exactly  $(q-1)/2$  triples in  $T$ ,*
- (3): *any even element from  $\{1, \dots, q\}$  occurs in exactly  $(q-3)/2$  triples in  $T$ .*

**Claim 7.4.** *If  $q$  is even, then*

- (1): *the element  $q+1$  occurs in exactly  $q/2$  triples in  $T$ ,*
- (2): *any element from  $\{1, \dots, q\}$  occurs in exactly  $(q-2)/2$  triples in  $T$ .*

If  $q$  is odd then there is an irreducible representation parametrised by  $u = (q+1)/2$ . In this case, and using Claim 7.3, the value of the sum in Eq. (7.1) is  $-(q+1)/2$  and by Eq. (2.4) the corresponding eigenvalue is  $-1$ . If  $u \neq \frac{q+1}{2}$ , then, using Claim 7.3 or Claim 7.4 as appropriate, the sum in Eq. (7.1) is 1. A straight-forward calculation using Eq. (2.4) then shows that the eigenvalue for this representation is  $2/(q-1)$ .  $\square$

**Lemma 7.5.** *Assume that  $\gcd(3, q+1) = 3$ . The eigenvalue for irreducible representations with degree  $q^2 - q + 1$  is equal to  $6q/(q^2 - q + 4)$ .*

*Proof.* Define

$$T' = \{(k, l, m) : k + l + m \equiv 0 \pmod{q+1}, 1 \leq k < l \leq (q+1)/3, \ell < m \leq q+1\}.$$

The conjugacy classes in family  $C'_2$ , when  $\gcd(3, q+1) = 3$ , are parametrised by a triple from  $T'$ .

Let  $\chi_u$  be an irreducible representation with degree  $q^2 - q + 1$ ; these characters are parameterized by  $u \in \{1, \dots, (q+1)/3 - 1\}$ . The value of  $\chi_u$  on the conjugacy

classes of type  $C_1$  is 0 and the value of  $\chi_u$  on the conjugacy class of type  $C_2''$  is 3. The sum of  $\chi_u$  over all conjugacy classes of type  $C_2'$  is

$$(7.2) \quad \sum_{(k,\ell,m) \in T'} e^{3ku} + e^{3\ell u} + e^{3mu},$$

(here  $e$  is a complex primitive  $(q+1)$ th root of unity).

The exact value of the sum in Eq. (7.2) can be determined, we will do this first for  $q$  odd and then for  $q$  even. If  $q$  is odd, then for every  $i$  there are, in total,  $\frac{q-2}{6}$  elements  $j$  in triples from  $T'$  with  $j \equiv 3i \pmod{q+1}$ . In this case we have that the sum in Eq. (7.2) is equal to

$$\frac{q-2}{3} \sum_{i=1}^{\frac{q+1}{3}} e^{3i} + \frac{q-2}{6} \sum_{i=1}^{\frac{q+1}{3}} e^{3i} = 0.$$

If both  $q$  and  $i$  are even, then there are in total  $\frac{q-5}{6}$  elements  $j$  in triples from  $T'$  for which  $j \equiv 3i \pmod{q+1}$ . If  $q$  is even and  $i$  is odd, then there are in total  $\frac{q+1}{6}$  elements  $j$  in triples from  $T'$  for which  $j \equiv 3i \pmod{q+1}$ . Thus for  $q$  even the sum in Eq. (7.2) is

$$\frac{q-2}{3} \sum_{i=1}^{\frac{q+1}{3}} e^{3i} + \frac{q-5}{6} \sum_{i=1}^{\frac{q+1}{6}} e^{6i} + \frac{q+1}{6} \sum_{i=1}^{\frac{q+1}{6}} e^{3(2i-1)} = 0.$$

A straight-forward application of Eq. (2.4) shows that the eigenvalue for these representations is  $6q/(q^2 - q + 4)$ .  $\square$

**Proposition 7.6.** *Let  $G = \text{PSU}_3(q)$ , then Theorem 1.1 holds.*

*Proof.* Let  $A$  be the  $(a, b)$ -weighted adjacency matrix defined above. The eigenvalues of  $A$  can be directly calculated for  $q \leq 5$ , so we will assume that  $q \geq 6$ . From Lemma 7.1,  $\lambda(\psi) = -1$ , we will show that this is the least eigenvalue of  $A$ .

If  $\chi$  is an irreducible representation with  $\lambda(\chi) \leq \lambda(\psi)$ , then by Lemma 2.6

$$\begin{aligned} \chi(1) &\leq (|\Omega| - 1) \sqrt{|G| \frac{\sum_{i=0}^{\ell} a_i^2 |C_i|}{(\sum_{i=1}^{\ell} a_i |C_i|)^2}} - 2 \\ &= (q^3) \sqrt{|G| \frac{(\frac{q(2q^2+q-1)}{3})^2 \frac{1}{|C_1|} + (\frac{q(q^2-q+1)}{3})^2 \frac{1}{|C_2|}}{(\frac{q(2q^2+q-1)+q(q^2-q+1)}{3})^2}} - 2. \end{aligned}$$

Some rather tedious, but not complicated, calculation show that this is strictly smaller than  $(q-1)(q^2 - q + 1)$  for  $q \geq 6$  (the cases where  $\gcd(3, q+1)$  is equal to 1 and 3 need to be considered separately).

The only irreducible representations of  $\text{PSU}_3(q)$  with degree less than  $(q-1)(q^2 - q + 1)$  are the representations with degree  $q(q-1)$  and  $q^2 - q + 1$  (this is from [30, Table 2]). We have calculated in Lemmas 7.1, 7.2 and 7.5 that the eigenvalue for any of these representations is one of  $-1$ ,  $2/(q-1)$  or  $6q/(q^2 - q + 4)$ . This implies that the least eigenvalue of  $A$  is  $-1$  and by Lemma 2.5 and intersecting set in  $G$  is no larger than  $|G|/(q^3 + 1)$ .  $\square$

## 8. PROJECTIVE LINEAR GROUPS: LINE 2 OF TABLE 1

By the reductions given in Section 3, we only need to consider the groups  $\text{PSL}_n(q)$ . Throughout this section, let  $G = \text{PSL}_n(q)$ . This group has  $\frac{\prod_{i=1}^{n-1} (q^n - q^i)}{d(q-1)}$  elements, where  $d = \gcd(n, q-1)$ , and a natural action of order  $\frac{q^n-1}{q-1}$  on the points

of the projective space. The first step in this proof is to show that the proportion of derangements in  $\text{PSL}_n(q)$  is large.

Given a positive integer  $n$ , we denote by  $\varphi(n)$  the *Euler totien function*.

**Lemma 8.1.** *Let  $n$  be a positive integer with  $n > 6$ . Then  $\varphi(n) \geq n/\log_2(n)$ , and  $\varphi(n) \geq 2n/(\log_3(n) + 2)$  if  $n$  is odd.*

*Proof.* Write  $n = 2^{\alpha_0} p_1^{\alpha_1} \cdots p_\ell^{\alpha_\ell}$  where  $\alpha_0, \ell \geq 0$  and  $p_1, \dots, p_\ell$  are distinct odd primes with  $p_1 < p_2 < \cdots < p_\ell$  and  $\alpha_1, \dots, \alpha_\ell \geq 1$ . Also write  $m = n/2^{\alpha_0}$ . Observe that  $\ell \leq \log_3(m)$ .

Assume first that  $\ell = 0$ , that is,  $n = 2^{\alpha_0}$  is a 2-power. Observe that  $\alpha_0 \geq 3$  because  $n > 6$ . Then  $\varphi(n) = n/2 \geq n/\alpha_0 = n/\log_2(n)$ . Assume next that  $n$  is odd, that is,  $\alpha_0 = 0$  and  $n = m$ . Now,

$$\begin{aligned} \varphi(n) &= n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_\ell}\right) \geq n \cdot \frac{2}{3} \cdot \frac{3}{4} \cdots \frac{\ell+1}{\ell+2} \\ &= \frac{2n}{\ell+2} \geq \frac{2n}{\log_3(n) + 2} \geq \frac{n}{\log_2(n)}, \end{aligned}$$

where the last inequality follows from an easy computation.

Finally assume that  $\alpha_0 > 0$  and  $\ell > 0$ . Now,

$$\varphi(n) = 2^{\alpha_0-1} \varphi(m) \geq 2^{\alpha_0} \frac{m}{\log_3(m) + 2} = \frac{n}{\log_3(m) + 2}.$$

If  $\alpha_0 \geq 2$ , then  $\log_3(m) + 2 \leq \log_2(m) + \alpha_0 = \log_2(n)$ . Suppose that  $\alpha_0 = 1$ . Now, if  $n \geq 14$ , then we get  $\log_3(m) + 2 = \log_3(n/2) + 2 \leq \log_2(n)$  and the lemma follows. If  $n = 10$ , then the lemma follows with a direct computation.  $\square$

Let  $n$  be a positive integer and let  $q$  be a prime power.

**Lemma 8.2.** *The proportion of derangements in  $\text{PSL}_n(q)$  is at least  $\frac{1}{n^2 \log_2(q)}$ .*

*Proof.* In the next paragraph we recall some information on the conjugacy classes of  $\text{PSL}_n(q)$ , which can be found for instance in [9, 10].

Let  $p$  be the proportion of derangements of  $G$  in its natural action on the projective space and let  $C$  be a Singer cycle of  $G$ . Now,  $C$  is a cyclic group of order  $\frac{q^n-1}{d(q-1)}$ , where  $d = \gcd(n, q-1)$ . Moreover,  $C = \mathbf{C}_G(C)$ ,  $|\mathbf{N}_G(C) : \mathbf{C}_G(C)| = n$  and  $\mathbf{N}_G(C)/\mathbf{C}_G(C)$  is cyclic and generated by an element of the Weyl group of  $G$ . Every non-identity element of  $C$  acts fixed-point-freely. Moreover, for every  $x \in C$  with  $C = \langle x \rangle$ , we have  $C = \mathbf{C}_G(x)$ .

Fix

$$D = \bigcup_{\substack{x \in C \\ C = \langle x \rangle}} x^G.$$

Now  $p \geq |D|/|G|$  and, from the previous paragraph,  $D$  is the union of at least  $\varphi(|C|)/n$   $G$ -conjugacy classes and each conjugacy class has size  $|G : C|$ . If  $|C| > 6$ , using Lemma 8.1, we obtain

$$\begin{aligned} p &\geq \frac{\frac{\varphi(|C|)}{n} \cdot |G : C|}{|G|} \geq \frac{1}{n \log_2(|C|)} \\ &\geq \frac{1}{n \log_2((q^n - 1)/(q - 1))} \geq \frac{1}{n \log_2(q^n)} = \frac{1}{n^2 \log_2(q)}. \end{aligned}$$

If  $|C| \leq 6$ , then  $n = 2$  and  $q \leq 11$ . For each of these groups we can check (with a case-by-case explicit computation) that the statement of the lemma holds.  $\square$

In light of recent results by Fulman and Guralnick [15, 16, 17, 18], Lemma 8.2 is rather weak (but still suitable for our application). In fact, answering a conjecture due independently to Shalev and Boston et al. [6], Fulman and Guralnick have proved that there exists a constant  $C > 0$  such that, in every non-abelian simple transitive permutation group, the proportion of elements which are derangements is at least  $C$ . In particular, in view of this remarkable theorem, in Lemma 8.2 one might replace the function  $1/n^2 \log_2(q)$  by the constant  $C$ . However, even for rather natural transitive actions, currently there is no good estimate on  $C$ : in fact, we are not aware of any bound uniform in  $q$  and  $n$  for the proportion of derangements in  $\text{PSL}_n(q)$  in its natural action on the projective space. Following the arguments in the work of Fulman and Guralnick this seems possible to achieve (at least for  $q \geq 7$ ), but it would take us too far astray to do it here.

**Proposition 8.3.** *Let  $G$  be a 2-transitive group as in line 2 of Table 1. Then Theorem 1.1 holds for  $G$ .*

*Proof.* From our preliminary reductions we may assume that  $G = \text{PSL}_n(q)$  is endowed of its natural action on the points of the projective space. Moreover, from the work in [28, 29], we may assume that  $n \geq 4$ .

Let  $\pi$  be the permutation character of  $G$  and write  $\pi = \chi_0 + \psi$ , where  $\chi_0, \psi$  are the irreducible constituents of  $\pi$  and  $\chi_0$  is the principal character of  $G$ . We show that  $\lambda(\psi)$  is the minimal eigenvalue of  $\Gamma_G$  and  $\psi$  is the only irreducible complex character of  $G$  realising this minimum. We argue by contradiction and we assume that there exists  $\chi^* \in \text{Irr}(G)$  with  $\chi^* \neq \psi$  and with  $\lambda(\chi^*) \leq \lambda(\psi)$ . From Lemmas 2.4 and 8.2, we have

$$(8.1) \quad \chi^*(1) \leq \frac{q^n - q}{q - 1} \sqrt{n^2 \log_2(q) - 2}.$$

We now use some character-theoretic results of the third author and Zalesskii [32]. For the reader's convenience we reproduce the relevant result from [32] when  $n \geq 4$ .

Denote by  $1 = d_0 < d_1 < d_2 < \dots < d_\ell$  the character degrees of  $G$  and by  $N_j$  the number of irreducible complex representations of  $G$  (up to equivalence) of degree  $d_j$ . (The definition of equivalence in this context is in [32] or in [23].) The value of  $(d_i, N_i)$  for  $i = 1, 2, 3$  are in Table 4.

$(d_1, N_1)$	$(d_2, N_2)$	$(d_3, N_3)$	Condition
$\left(\frac{q^4 - q}{q - 1}, 1\right)$	$\left(\frac{q^4 - 1}{q - 1}, q - 2\right)$	$(\frac{1}{2}(q^3 - 1)(q - 1), 2)$	$n = 4, 2 \nmid q, q \neq 3$
$\left(\frac{q^4 - q}{q - 1}, 1\right)$	$\left(\frac{q^4 - 1}{q - 1}, q - 2\right)$	$((q^3 - 1)(q - 1), q/2)$	$n = 4, 2 \mid q, q \neq 2$
$(7, 1)$	$(8, 1)$	$(14, 1)$	$(n, q) = (4, 2)$
$(26, 2)$	$(39, 1)$	$(40, 1)$	$(n, q) = (4, 3)$
$\left(\frac{q^n - q}{q - 1}, 1\right)$	$\left(\frac{q^n - 1}{q - 1}, q - 2\right)$	$\left(\frac{(q^n - 1)(q^{n-1} - q^2)}{(q - 1)(q^2 - 1)}, 1\right)$	$n \geq 5, q \geq 3, (n, q) \neq (6, 3)$
$\left(\frac{q^n - q}{q - 1}, 1\right)$	$\left(\frac{(q^n - 1)(q^{n-1} - q^2)}{(q - 1)(q^2 - 1)}, 1\right)$	$\left(\frac{(q^n - 1)(q^{n-1} - 1)}{(q - 1)(q^2 - 1)}, 1\right)$	$n \geq 5, n \neq 6, q = 2$
$(62, 1)$	$(217, 1)$	$(588, 1)$	$(n, q) = (6, 2)$
$(363, 1)$	$(364, 1)$	$(6292, 2)$	$(n, q) = (6, 3)$

TABLE 4. Small character degrees of  $\text{PSL}_n(q)$

Comparing Eq. (8.1) with Table 4, we are left with one of the following cases:

- (1):  $q > 2, n \geq 5, \chi^*(1) = d_2$ ,
- (2):  $q = 2, n \in \{5, 6\}$ ,
- (3):  $q \geq 16, q \neq 17, n = 4$  and  $\chi^*(1) = d_2$ ,
- (4):  $q \in \{2, 3, 4, 5, 7, 8, 9, 11, 13, 17\}$  and  $n = 4$ .

Now, with the computer algebra system **Magma** [5], we can check directly that in Cases (2) and (4) the minimum eigenvalue is  $\lambda(\psi)$  and is realised only by  $\psi$ . It remains to consider Cases (1) and (3). Here,  $\chi^*$  is one of the irreducible characters of  $G$  of degree  $(q^n - 1)/(q - 1)$ . These characters are named Weil characters, after the pioneering work of André Weil in [35]. An extensive study of Weil characters has begun in [19] and now the value of each Weil character in each conjugacy class of  $G$  is explicitly known. For example it is given in the account of Guralnick and the third author (see Eq. (1) on page 4976 in [24].) From this formula it is immediate to see that  $\lambda(\chi^*) = 0$ , contradicting the fact that  $\lambda(\chi^*) \leq \lambda(\psi)$ .  $\square$

The proof of Proposition 8.3 can be slightly shortened and simplified using a suitable weighted adjacency matrix and using the method we present in the next section for dealing with the Symplectic groups.

### 9. SYMPLECTIC GROUPS: LINES 3 AND 4 OF TABLE 1

The proof of Theorem 1.1 for the two 2-transitive actions of the symplectic group  $\mathrm{Sp}_{2n}(2)$  is similar to a combination of the proofs in the projective linear and unitary cases. The investigations of Guralnick and the third author in [24] on the irreducible complex characters of  $\mathrm{Sp}_{2n}(2)$  of “small” degree will be crucial to this proof. Details about the symplectic group can be found in [9, 10]. We start by setting some notation.

We fix  $n$  to be a natural number with  $n \geq 3$ . We let  $G = \mathrm{Sp}_{2n}(2)$  and we let  $V$  be the  $2n$ -dimensional vector space over the finite field  $\mathbb{F}_2$ . (Thus  $V$  is the natural module for  $G$ .) The group  $G$  has two natural 2-transitive actions. Both actions can be seen by viewing  $\mathrm{Sp}_{2n}(2) = \Omega_{2n+1}(2)$ , and recalling that  $\Omega_{2n+1}(2)$  has a natural action on the non-degenerate hyperspaces of plus type (which we denote by  $\Omega^+$ ) and on the non-degenerate hyperspaces of minus type (which we denote by  $\Omega^-$ ). We have

$$|\Omega^+| = 2^{n-1}(2^n + 1), \quad |\Omega^-| = 2^{n-1}(2^n - 1).$$

Moreover, the stabiliser of an element of  $\Omega_n^+$  is  $H^+ = \mathrm{O}_{2n}^+(2)$  and the stabiliser of an element of  $\Omega_n^-$  is  $H^- = \mathrm{O}_{2n}^-(2)$ .

Given  $\varepsilon \in \{+, -\}$ , we denote by  $\pi^\varepsilon$  the permutation character of  $G$  in its action on  $\Omega^\varepsilon$  and by  $\mathcal{D}^\varepsilon$  the set of derangements of  $G$  in its action on  $\Omega^\varepsilon$ . We write

$$(9.1) \quad \pi^+ = \chi_0 + \rho^+, \quad \pi^- = \chi_0 + \rho^-,$$

where  $\chi_0$  is the principal character of  $G$ . (Later in our arguments we will be using [24]; hence we point out that the characters  $\rho^+$  and  $\rho^-$  are studied in [24, Section 6] in which  $\rho^+$  is denoted by  $\rho_n^2$ , and  $\rho^-$  is denoted by  $\rho_n^1$ .)

We recall that  $\chi_0 + \rho^+ + \rho^-$  is the permutation character for the transitive action of  $G$  on the non-zero elements of  $V$  (see for example [24], or [22] for a combinatorial proof), and hence

$$(9.2) \quad (\rho^+ + \rho^-)(g) = 2^{\dim \mathrm{Ker}(g-1_V)} - 2$$

for every  $g \in G$ .

In the next paragraph we recall some information on the conjugacy classes of  $G = \mathrm{Sp}_{2n}(2)$ , we again refer the reader to [9, 10].

If  $\varepsilon = +$ , then  $\varepsilon \cdot 1$  represents 1; similarly, if  $\varepsilon = -$ , then  $\varepsilon \cdot 1$  represents  $-1$ . Embedding

$$T^\varepsilon = \Omega_2^{-\varepsilon}(2^n) < \mathrm{SL}_2(2^n) \cong \mathrm{Sp}_2(2^n)$$

naturally in  $G = \mathrm{Sp}_{2n}(2)$  (by identifying  $W = \mathbb{F}_{2^n}^2$  with  $V = \mathbb{F}_2^{2n}$ ), we see that  $G$  contains a cyclic maximal torus  $T^\varepsilon$  of order  $2^n + \varepsilon \cdot 1$ . Fix a generator  $x^\varepsilon$  of  $T^\varepsilon$ .



We claim that that  $x^\varepsilon$  is contained in a unique conjugate of  $H^{-\varepsilon}$  and  $x^\varepsilon \in \mathcal{D}^\varepsilon$ ; equivalently,

$$(9.3) \quad \rho^\varepsilon(x^\varepsilon) = -1, \quad \rho^{-\varepsilon}(x^\varepsilon) = 0.$$

This can be seen as follows. By construction,  $x^\varepsilon$  has no nonzero fixed points on  $W$  and so on  $V$  as well. Hence Eq. (9.2) implies that  $\rho^+(x^\varepsilon) + \rho^-(x^\varepsilon) = -1$ . On the other hand,  $\pi^+(x^\varepsilon) = 1 + \rho^+(x^\varepsilon)$  is the number of  $x^\varepsilon$ -fixed points on  $\Omega^+$ , and so  $\mathbb{Z} \ni \rho^+(x^\varepsilon) \geq -1$ ; similarly,  $\mathbb{Z} \ni \rho^-(x^\varepsilon) \geq -1$ . It follows that

$$\{\rho^+(x^\varepsilon), \rho^-(x^\varepsilon)\} = \{0, -1\}.$$

Also note that if  $\varepsilon = -$  then  $T^- = \Omega_2^+(2^n)$  can be embedded in  $H^+ = \Omega_{2n}^+(2)$  again by base change, so  $\rho^+(x^-) \neq -1$ . Next, for  $n > 3$  let  $p$  be a *primitive prime divisor* of  $2^{2n} - 1$  (cf. [36]), so that  $p \mid (2^n + 1) = |x^+|$  but  $p \nmid |H^+|$ . It follows for  $n > 3$  that  $x^+$  cannot be contained in any conjugate of  $H^+$ , whence  $\rho^+(x^+) = -1$ . The same holds for  $n = 3$  by inspecting [11]. Consequently, Eq. (9.3) holds for all  $n \geq 3$ .

We also note that  $T^\varepsilon = \mathbf{C}_G(T^\varepsilon) = \mathbf{C}_G(x^\varepsilon)$  and hence  $|\mathbf{C}_G(x^\varepsilon)| = 2^n + \varepsilon \cdot 1$ .

Let  $A^\varepsilon$  be the square matrix indexed by the elements of  $G$  with

$$(A^\varepsilon)_{g,h} = \begin{cases} 1 & \text{if } gh^{-1} \in (x^\varepsilon)^G, \\ 0 & \text{if } gh^{-1} \notin (x^\varepsilon)^G. \end{cases}$$

As  $(x^\varepsilon)^G \subseteq \mathcal{D}^\varepsilon$ , we see that  $A^\varepsilon$  is a weighted adjacency matrix for the derangement graph of  $G$  in its action on  $\Omega^\varepsilon$ .

**Proposition 9.1.** *Let  $G$  be a 2-transitive group as in line 3 or 4 of Table 1. Then Theorem 1.1 holds for  $G$ .*

*Proof.* We can check with a computer that this holds when  $n \leq 6$ . Therefore we assume that  $n \geq 7$ .

We use the notation that we have established above. Moreover, given  $\chi \in \text{Irr}(G)$ , we denote by  $\lambda(\chi, \varepsilon)$  the eigenvalue of  $A^\varepsilon$  afforded by  $\chi$  (see Eq. (2.4)). We determine the maximum and the minimum eigenvalue of  $A^\varepsilon$ . Since all the weights are non-negative real numbers, the largest eigenvalue  $d^\varepsilon$  of  $A^\varepsilon$  is realised by the principal character of  $G$ . Therefore

$$(9.4) \quad d^\varepsilon = \lambda(\chi_0, \varepsilon) = |(x^\varepsilon)^G| = \frac{|G|}{|\mathbf{C}_G(x^\varepsilon)|} = \frac{|G|}{2^n + \varepsilon \cdot 1}.$$

Let  $\tau^\varepsilon$  be the minimum eigenvalue of  $A^\varepsilon$ . We prove that  $\tau^\varepsilon = \lambda(\rho^\varepsilon, \varepsilon)$  and that  $\rho^\varepsilon$  is the unique character affording  $\tau^\varepsilon$ . We argue by contradiction and we assume that there exists  $\chi^*$  with  $\chi^* \neq \rho^\varepsilon$  and  $\lambda(\chi^*, \varepsilon) \leq \lambda(\rho^\varepsilon, \varepsilon)$ .

Lemma 2.6 gives

$$(9.5) \quad \begin{aligned} \chi^*(1) &\leq (|\Omega^\varepsilon| - 1) \sqrt{|G| \frac{|(x^\varepsilon)^G|}{|(x^\varepsilon)^G|^2} - 2} \\ &= (|\Omega^\varepsilon| - 1) \sqrt{|\mathbf{C}_G(x^\varepsilon)| - 2} = (|\Omega^\varepsilon| - 1) \sqrt{2^n + \varepsilon \cdot 1 - 2}. \end{aligned}$$

The main result of Guralnick and the third author [24, Theorem 1.1 and Table 1] shows that there exist irreducible characters  $\alpha_n, \beta_n$  and  $\zeta_n^1$  with

$$\alpha_n(1) = \frac{(2^{n-1} - 1)(2^n - 1)}{3}, \quad \beta_n(1) = \frac{(2^{n-1} + 1)(2^n + 1)}{3}, \quad \zeta_n^1(1) = \frac{2^{2n} - 1}{3}$$

such that, if  $\chi \in \text{Irr}(G)$  and  $\chi(1) > 1$ , then either

- (1):  $\chi \in \{\rho^+, \rho^-, \alpha_n, \beta_n, \zeta_n^1\}$ , or
- (2):  $\chi(1) \geq ((2^{n-1} + 1)(2^{n-2} - 2)/3 - 1)2^{n-2}(2^{n-1} - 1)$ .

We have

$$\left( \frac{(2^{n-1} + 1)(2^{n-2} - 2)}{3} - 1 \right) 2^{n-2}(2^{n-1} - 1) > (|\Omega^\varepsilon| - 1)\sqrt{2^n + \varepsilon \cdot 1 - 2}$$

and hence, by Eq. (9.5), we have  $\chi^* \in \{\rho^+, \rho^-, \alpha_n, \beta_n, \zeta_n^1\}$ .

From Eq. (2.4), we have

$$(9.6) \quad \lambda(\rho^\varepsilon, \varepsilon) = \frac{-|(x^\varepsilon)^G|}{|\Omega^\varepsilon| - 1} = -\frac{|G|}{(2^n + \varepsilon \cdot 1)(|\Omega^\varepsilon| - 1)}.$$

Now, Eq. (9.3) yields  $\lambda(\rho^{-\varepsilon}, \varepsilon) \geq 0 > \lambda(\rho^\varepsilon, \varepsilon)$  and hence  $\chi^* \in \{\alpha_n, \beta_n, \zeta_n^1\}$ .

Recall that  $V = \mathbb{F}_2^{2n}$  is the natural module for  $G$ . Observe that by taking the tensor product  $V \otimes_{\mathbb{F}_2} \mathbb{F}_4$  we may view  $V$  as a  $2n$ -dimensional vector space over  $\mathbb{F}_4$ . Let  $\xi$  be a primitive third root of unity in  $\mathbb{F}_4$  and let  $\bar{\xi}$  be a primitive third root of unity in  $\mathbb{C}$ . Clearly, 1,  $\xi$  and  $\xi^2$  are not eigenvalues of the matrices  $x^+$  and  $x^-$  in their action on  $V$  and hence  $\text{Ker}(x^\varepsilon - 1_V) = \text{Ker}(x^\varepsilon - \xi \cdot 1_V) = \text{Ker}(x^\varepsilon - \xi^2 \cdot 1_V) = 0$ . Now it follows from [24, page 4997, Eq. (4)] that

$$(9.7) \quad \zeta_n^1(x^\varepsilon) = \frac{1}{3}(1 + \bar{\xi} + \bar{\xi}^2) = 0.$$

Therefore from Eq. (2.4) we have  $\lambda(\zeta_n^1, \varepsilon) = 0 > \lambda(\rho^\varepsilon, \varepsilon)$  and hence  $\chi^* \in \{\alpha_n, \beta_n\}$ .

Consider the map  $\zeta_n : G \rightarrow \mathbb{C}$  defined by  $\zeta_n(g) = (-2)^{\dim \text{Ker}(g-1_V)}$ . It turns out (see [24, pages 4976, 4977]) that  $\zeta_n$  is a character of  $G$  and

$$(9.8) \quad \zeta_n = \alpha_n + \beta_n + 2\zeta_n^1.$$

Next, consider the map  $\kappa_\varepsilon : \text{O}_{2n}^\varepsilon(2) \rightarrow \{-1, 1\}$  defined by  $\kappa_\varepsilon(g) = (-1)^{\dim \text{Ker}(g-1_V)}$ . Now,  $\kappa_\varepsilon$  is actually a homomorphism (see for example [11, page xii]) with kernel the index 2 subgroup  $\Omega_{2n}^\varepsilon(2)$  of  $\text{O}_{2n}^\varepsilon(2)$ , and hence  $\kappa_\varepsilon$  is a character of  $H^\varepsilon = \text{O}_{2n}^\varepsilon(2)$ . It is shown in [24, Eq. (11)] that

$$(9.9) \quad \text{Ind}_{H^+}^G(\kappa_+) + \text{Ind}_{H^-}^G(\kappa_-) = \zeta_n.$$

Note that

$$\alpha_n(1) + \beta_n(1) < \alpha_n(1) + \zeta_n^1(1) = [G : H^-] < [G : H^+] = \beta_n(1) + \zeta_n^1(1) < 2\zeta_n^1(1).$$

Together with Eqs. (9.8) and (9.9), this implies that

$$\text{Ind}_{H^+}^G(\kappa_+) = \beta_n + \zeta_n^1, \quad \text{Ind}_{H^-}^G(\kappa_-) = \alpha_n + \zeta_n^1.$$

Certainly,  $\text{Ind}_{H^+}^G(\kappa_+)$  is zero on every element not conjugate to an element of  $H^+$  (and hence on  $\mathcal{D}^+$ ), and similarly that  $\text{Ind}_{H^-}^G(\kappa_-)$  is zero on every element not conjugate to an element of  $H^-$  (and hence on  $\mathcal{D}^-$ ). It follows from Eq. (9.3) that

$$(\alpha_n + \zeta_n^1)(x^-) = (\beta_n + \zeta_n^1)(x^+) = 0$$

and hence, from Eq. (9.7), we obtain

$$(9.10) \quad \alpha_n(x^-) = 0, \quad \beta_n(x^+) = 0.$$

Therefore,  $\lambda(\alpha_n, -) = \lambda(\beta_n, +) = 0 > \lambda(\rho^\varepsilon, \varepsilon)$  and hence  $\chi^* = \alpha_n$  when  $\varepsilon = +$  and  $\chi^* = \beta_n$  when  $\varepsilon = -$ .

Now,  $\zeta_n(x^\varepsilon) = (-2)^{\dim \text{Ker}(x^\varepsilon - 1_V)} = (-2)^0 = 1$  and hence  $(\alpha_n + \beta_n)(x^\varepsilon) = 1$  by Eq. (9.7). Thus, Eq. (9.10) gives

$$\alpha_n(x^+) = 1, \quad \beta_n(x^-) = 1.$$

Therefore,  $\lambda(\alpha_n, +), \lambda(\beta_n, -) > 0 > \lambda(\rho^\varepsilon, \varepsilon)$ . This finally contradicts the existence of  $\chi^*$ .

Let  $S$  be an independent set of the derangement graph of  $G$  in its action on  $\Omega^\varepsilon$ . Now, using Lemma 2.5 and Eqs. (9.4) and (9.6), we obtain

$$|S| \leq |G| \left(1 - \frac{d^\varepsilon}{\tau^\varepsilon}\right)^{-1} = \frac{|G|}{|\Omega^\varepsilon|}$$

and the proposition is proven.  $\square$

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